Evaluating Integrals (Section 5.3) and the Fundamental Theorem of Calculus (Section (5.4)

Intro to 5.3

Today we'll go through the math of how to evaluate integrals using antiderivatives.

Evaluation Theorem

The antiderivative gives an easy way to evaluate definite integrals. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Evaluation Theorem

This theorem is a big deal!

- Adding up millions of tiny rectangles under a curve
- Evaluating an antiderivative of a function

These turn out to be the same thing!

Evaluation Theorem

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Where does the $+C$ go?

Evaluation Theorem

New notation:

$$F(x)\Big|_a^b = F(b) - F(a)$$

Indefinite Integrals

New notation: a new way to write the **antiderivative**.

$$\int f(x)\,dx = F(x)$$

This is called an indefinite integral. No limits of integration.
From now on, the phrases “the indefinite integral” and “the antiderivative” are interchangeable.

Indefinite Integrals

We've seen almost all of these before.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad \int cf(x)\,dx = c\int f(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \boxed{\int a^x\,dx = \frac{a^x}{\ln a} + C}$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

Example

Evaluate the definite integral:

$$\int_{-2}^{3} 12x^2 + 5\,dx$$

Example

Evaluate the definite integral:

$$\int_0^2 e^x + \frac{x^3 + x^4}{x^2}\, dx$$

Try it!

Evaluate the definite integral:

$$\int_4^9 \frac{1}{\sqrt{x}} - e^x - 1 \, dx$$

Example

Evaluate the definite integral:

$$\int_{-5}^{5} |x| \, dx$$

Hint: $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

Applications

If we start with a function which is already a derivative, then taking the antiderivative just gives us the original function back.
So:

$$\int_a^b v(t)\,dt = s(b) - s(a)$$

This gives the net change in position.

Applications

The same concept works with any rate of change.

- Integral of rate of water flowing into a pool equals total change in volume of water
- Integral of rate of blood flow through a vein equals total amount of blood pumped through
- Integral of rate of change of population equals net change in population

Applications

Because of conservation efforts, the population of bald eagles is increasing. Suppose the population has rate of change equal to

$$v(t) = 30t^2 + 100t + 10 \text{ eagles/year}$$

What will be the net increase in the bald eagle population in 10 years?

The fundamental theorem of calculus: Intro

The fundamental theorem of calculus describes the exact way in which the integral and the derivative are opposite operations. We will go over several applications as well.

Function defined with an integral

Look at this function:

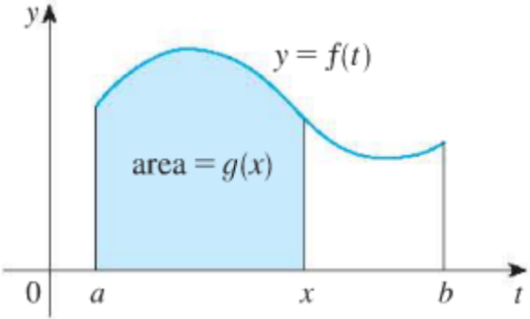
$$g(x) = \int_0^x f(t)\,dt$$

This is a **function of** x

Function defined with an integral

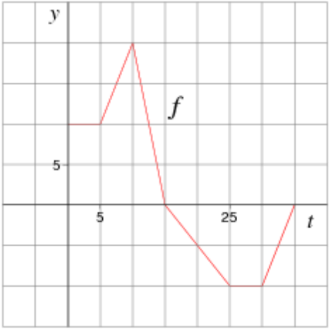
$$g(x) = \int_0^x f(t)\, dt$$

Think of the function in terms of area. As x increases we pick up more area under of the function $f(t)$.



Example

Find $g(10)$ and $g(25)$ for $g(x) = \int_0^x f(t)\,dt$



The fundamental theorem of calculus

The fundamental theorem of calculus has two parts.

- (Evaluation theorem from earlier) If $F(x)$ is an antiderivative of $f(x)$, then
$$\int_a^b f(x)\,dx = F(b) - F(a)$$
- If the function $f(x)$ is continuous, then
$$\left(\int_a^x f(t)dt\right)' = f(x)$$

  The letter a stands for a constant number. This formula only holds with just plain x.

The fundamental theorem of calculus

- (Evaluation theorem from earlier) If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- If the function $f(x)$ is continuous, then

$$\left(\int_a^x f(t)dt\right)' = f(x)$$

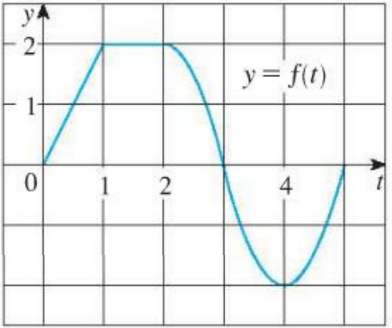
  The letter a stands for a constant number. This formula only holds with just plain x.

How do these show that the integral and derivative are opposite operations?

Example

For the function $g(x) = \int_0^x f(t)\,dt$, find

- The intervals where $g(x)$ is increasing/decreasing
- The intervals where $g(x)$ is concave up/down



Example

Find the derivative of $g(x)$ for

$$g(x) = \int_5^x \sin(t^2)\,dt$$

Try it!

Find the derivative of $g(x)$ for

$$g(x) = \int_{-3}^{x} \frac{\ln(t) - 6t^2}{e^t + 7}\, dt$$

Example

Use both methods to find the derivative of $g(x)$.

$$g(x) = \int_0^x 4t - 5\,dt$$

Example

Find the derivative of $g(x)$.

$$g(x) = \int_0^{3x} 5t^2 - 2e^{5t}\, dt$$

Example

Find the derivative of $g(x)$.

$$g(x) = \int_{x}^{5x} \sin(t) + 2t^2 \, dt$$

Try it!

Find the derivative of $g(x)$.

$$g(x) = \int_{x}^{x^2} e^{3t}\, dt$$